THE FREUDENTHAL-SPRINGER-TITS CONSTRUCTIONS OF EXCEPTIONAL JORDAN ALGEBRAS

BY
KEVIN McCRIMMON

Constructions of 27-dimensional exceptional simple Jordan algebras have been given by H. Freudenthal [2], [3], T. A. Springer [8], and J. Tits [9]. In the first two approaches the cubic generic norm form plays a central role, with applications to projective geometry and algebraic groups; the third approach gives a simple method for constructing all exceptional simple algebras. The constructions are limited to fields of characteristic $\neq 2$ as usual for Jordan algebras defined in terms of a bilinear multiplication, and in order to polarize the cubic norm form the characteristic must be $\neq 3$.

Recently a definition of Jordan algebras has been proposed [5] which is based on a cubic composition involving U-operators. A *unital Jordan algebra* over a commutative associative ring Φ is a triple $\mathfrak{J}=(\mathfrak{X}, U, 1)$ where $\mathfrak{X}$ is a Φ-module, U a quadratic mapping $x \to U_x$ of $\mathfrak{X}$ into $\mathrm{Hom}_\Phi(\mathfrak{X}, \mathfrak{X})$, and 1 an element of $\mathfrak{X}$ satisfying the axioms

(1) $U_1=1$,

(2) $U_{U(x)y}=U_xU_yU_x$,

(3) $U_x\{yxz\}=\{U_xyzx\}$.

($\{xyz\}=U_{x,z}y$ for $U_{x,z}=U_{x+z}-U_x-U_z$), and such that these hold under all scalar extensions (equivalently, the axioms can be linearized). For fields of characteristic $\neq 2$ this is equivalent to the usual definition of Jordan algebras (with $U_x=2L_x^2-L_{x^2}$).

In this paper the constructions of Freudenthal, Springer, and Tits will be derived as special cases of one general construction valid for all characteristics. The basic axioms used go back to [2]. It was Professor Springer who first pointed out that this approach could be carried out for arbitrary characteristics, and the author is indebted to him for suggesting the explicit formula for the Jordan structure.

1. **The general construction.** Let $\mathfrak{X}$ be a module over a commutative associative ring Φ. We assume we are given (i) a cubic form N on $\mathfrak{X}$ with values in Φ (so N is homogeneous of degree 3 and $N(x+\lambda y)=N(x)+\lambda\, \partial_yN|_x+\lambda^2\, \partial_xN|_y+\lambda^3N(y)$ where the differential $\partial_yN|_x$ of N at x in the direction y is linear in y and quadratic in x), (ii) a quadratic mapping $x \to x^\#$ in $\mathfrak{X}$, and (iii) a basepoint $c \in \mathfrak{X}$ related by

(4) $x^{\#\#}=N(x)x$ ("adjoint identity"),

(5) $N(c)=1$ ("basepoint identity"),

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(6) $T(x^{\#}, y) = \partial_y N|_x$ for $T(x, y) = -\partial_x \partial_y \log N|_c$,

(7) $c^{\#} = c$,

(8) $c \times y = T(y)c - y$,

and such that these hold under all scalar extensions of Φ; here the trace is $T(y) = T(y, c)$, we have a symmetric bilinear product by polarizing the adjoint

$$x \times y = (x+y)^{\#} - x^{\#} - y^{\#},$$

and the logarithmic derivative is

$$\partial_x \partial_y \log N|_z = N(z)^{-2}\{(N(z)\, \partial_x \partial_y N|_z - (\partial_x N|_z)(\partial_y N|_z)\}$$

whenever $N(z)$ is invertible (in particular, for $z = c$ if $N(c) = 1$).

We introduce a U-operator

(9) $U_x y = T(x, y)x - x^{\#} \times y$,

choose $1 = c$ as unit element, and denote by $\mathfrak{J}(N, \#, c)$ the triple $(\mathfrak{X}, U, c)$. We claim this is a Jordan structure. This is analogous to the result that if Q is a quadratic form on $\mathfrak{X}$ and $c \in \mathfrak{X}$ a basepoint where $Q(c) = 1$ then

$$U_x y = T(x, y)x - Q(x)y^*$$

defines a Jordan structure $\mathfrak{J}(Q, c)$ on $\mathfrak{X}$ (here $T(x, y) = Q(x, y^*)$, $y^* = Q(x, c)c - y$ [5, p. 1073]), called the Jordan algebra of the quadratic form Q.

We can immediately establish the first axiom (1) for $\mathfrak{J}(N, \#, c)$; indeed, (8) has been smuggled in just for this purpose:

$$U_c y = T(c, y)c - c \times y = T(y)c - c \times y = y$$

by (8) and (7).

The other axioms (2) and (3) are more troublesome. We will need several auxiliary formulas. We can linearize (4) by our assumption that the identities hold for all extensions, so

$$x^{\#} \times (x \times y) = N(x)y + T(x^{\#}, y)x, \tag{10}$$

$$x^{\#} \times y^{\#} + (x \times y)^{\#} = T(x^{\#}, y)y + T(y^{\#}, x)x, \tag{11}$$

$$x^{\#} \times (y \times z) + (x \times y) \times (x \times z) = T(x^{\#}, y)z + T(x^{\#}, z)y + T(y \times z, x)x. \tag{12}$$

Next, $c^{\#} = c$ and Euler's formula for the homogeneous form N give

$$T(x^{\#}, x) = \partial_x N|_x = 3N(x), \tag{13}$$

$$N(c) = 1, \qquad T(c) = 3. \tag{14}$$

Also

$$T(x \times y, z) = T(x, y \times z) \tag{15}$$

is symmetric in x, y, z since it equals $\partial_y T(x^\#, z)|_x = \partial_y \partial_z N|_x = \partial_x \partial_y \partial_z N|_w$ for any w since N is of degree 3. We have

$$T(x \times y) = T(x)T(y) - T(x, y), \tag{16}$$

since $T(x \times y, c) = T(x, y \times c) = T(y)T(x, c) - T(x, y)$ by (15) and (8), and

$$U_x(x \times y) = T(x^\#, y)x - N(x)y, \tag{17}$$

since $T(x, x \times y)x - x^\# \times (x \times y) = T(x \times x, y)x - N(x)y - T(x^\#, y)x$ and $x \times x = 2x^\#$ by (15) and (10). Furthermore,

$$x^\# \times x = [T(x^\#)T(x) - N(x)]c - T(x^\#)x - T(x)x^\#, \tag{18}$$

since

$$\begin{aligned} x^\# \times x - T(x)T(x^\#)c + T(x)x^\# &= x^\# \times [x - T(x)c] \\ &= -x^\# \times (x \times c) = -N(x)c - T(x^\#)x \end{aligned}$$

by (8) and (10). Finally we come to the most difficult formula

$$x \times (x^\# \times y) = N(x)y + T(x, y)x^\#. \tag{19}$$

We compute

$$\begin{aligned} 0 &= [x \times (x^\# \times y) + \{N(x)y + T(x^\#, y)x\} + y \times \{[T(x^\#)T(x) - N(x)]c - T(x^\#)x - T(x)x^\#\}] \\ &\qquad - [x \times (x^\# \times y) + x^\# \times (y \times x) + y \times (x \times x^\#)] \quad \text{(by (10), (18))} \\ &= [x \times (x^\# \times y) + N(x)y + T(x^\#, y)x + T(y)\{T(x^\#)T(x) - N(x)\}c \\ &\qquad - \{T(x^\#)T(x) - N(x)\}y - T(x^\#)y \times x - T(x)y \times x^\#] \\ &\quad - [\{T(x)T(x^\# \times y) + T(x^\#)T(y \times x) + T(y)T(x \times x^\#) - T(x \times x^\#, y)\}c \\ &\quad - T(x)x^\# \times y - T(x^\#)y \times x - T(y)x \times x^\# - T(x \times x^\#)y - T(x^\# \times y)x - T(y \times x)x^\#] \\ &\qquad\qquad \text{(by (8) and linearized (18))} \\ &= x \times (x^\# \times y) + \{2N(x) - T(x)T(x^\#) + T(x \times x^\#)\}y + \{T(x^\#, y) + T(x^\# \times y)\}x \\ &\quad + \{T(x \times y)\}x^\# + \{T(y)[T(x)T(x^\#) - N(x)] - 3T(x)T(x^\#)T(y) + T(x)T(x^\#, y) \\ &\qquad + T(x^\#)T(y, x) + T(y)T(x, x^\#) + [T(x)T(x^\#) - N(x)]T(c, y) \\ &\qquad - T(x^\#)T(x, y) - T(x)T(x^\#, y)\}c \\ &\qquad + T(y)\{[T(x)T(x^\#) - N(x)]c - T(x)x^\# - T(x^\#)x\} \quad \text{(by (16), (18))} \\ &= x \times (x^\# \times y) + \{2N(x) - T(x, x^\#)\}y - T(x, y)x^\# \\ &\qquad + \{3T(y)[T(x)T(x^\#) - N(x)] - 3T(x)T(x^\#)T(y) + T(y)T(x, x^\#)\}c \quad \text{(by (16))} \\ &= x \times (x^\# \times y) - N(x)y - T(x, y)x^\# \qquad \text{(by (13))}. \end{aligned}$$

(REMARK. This can be proved much more simply if Φ is a field, which we can assume is infinite. It suffices to prove (19) on the Zariski-dense set where $N(x)\neq 0$, and

$$N(x)\{x\times(x^\#\times y)\} = x^{\#\#}\times(x^\#\times y) = N(x^\#)y+T(x^{\#\#},y)x^\#$$
$$= N(x)^2y+N(x)T(x,y)x^\# = N(x)\{N(x)y+T(x,y)x^\#\},$$

where $N(x^\#)=N(x)^2$ since on the dense set $x^\#\neq 0$ we have $N(x^\#)x^\#=x^{\#\#\#}=\{N(x)x\}^\#=N(x)^2x^\#$ because $\#$ is quadratic.)

Now we are in a position to prove the axioms (2) and (3). For (2), using (9) we get

$$\begin{aligned}
&U_{U(x)y}z-U_xU_yU_xz\\
&\quad= T(U_xy,z)U_xy-(U_xy)^\#\times z-T(x,z)U_xU_yx+U_xU_y(x^\#\times z)\\
&\quad= \{T(x,y)T(x,z)-T(x^\#\times y,z)\}U_xy-\{T(x,y)x-x^\#\times y\}^\#\times z-T(x,z)\\
&\qquad\cdot\{T(x,y)U_xy-U_x(y^\#\times x)\}+T(y,x^\#\times z)U_xy-U_x\{y^\#\times(x^\#\times z)\}\\
&\quad= -T(x,y)^2x^\#\times z+T(x,y)\{x\times(x^\#\times y)\}\times z-(x^\#\times y)^\#\times z\\
&\qquad+T(x,z)\{T(x^\#,y^\#)x-N(x)y^\#\}-T(x,y^\#\times(x^\#\times z))x+x^\#\times(y^\#\times(x^\#\times z))\\
&\qquad\qquad\text{(by (15), (17), and since $\#$ is quadratic)}\\
&\quad= -T(x,y)^2x^\#+z+T(x,y)\{N(x)y+T(x,y)x^\#\}\times z\\
&\qquad-\{-x^{\#\#}\times y^\#+T(x^{\#\#},y)y+T(y^\#,x^\#)x^\#\}\times z+T(x,z)T(x^\#,y^\#)x\\
&\qquad-T(x,z)N(x)y^\#-T(x\times(x^\#\times z),y^\#)x+\{-(x\times y^\#)\times(x\times(x^\#\times z))\\
&\qquad+T(x^\#,y^\#)x^\#\times z+T(x^\#,x^\#\times z)y^\#+T(x,y^\#\times(x^\#\times z))x\}\\
&\qquad\qquad\text{(by (19), (11), (15), (12))}\\
&\quad= N(x)(x\times y^\#)\times z+T(x,z)T(x^\#,y^\#)x+T(x,z)N(x)y^\#-(x\times y^\#)\\
&\qquad\times\{N(x)z+T(x,z)x^\#\}\qquad\text{(by (4), (15), (19))}\\
&\quad= T(x,z)T(x^\#,y^\#)x+T(x,z)N(x)y^\#-T(x,z)x^\#\times(x\times y^\#)\\
&\quad= 0\qquad\text{(by (10)).}
\end{aligned}$$

For (3),

$$\begin{aligned}
&U_x\{yxz\}-\{U_xyzx\}\\
&\quad= U_xU_{y,z}x-U_{T(x,y)x,x}z+U_{x^\#\times y,x}z\qquad\text{(by (9))}\\
&\quad= U_x\{T(y,x)z+T(z,x)y-(y\times z)\times x\}-2T(x,y)U_xz+T(x^\#\times y,z)x\\
&\qquad+T(x,z)x^\#\times y-z\times\{x\times(x^\#\times y)\}\qquad\text{(by linearized (9))}\\
&\quad= -T(x,y)U_xz+T(x,z)U_xy-\{T(x^\#,y\times z)x-N(x)y\times z\}+T(x^\#\times y,z)x\\
&\qquad+T(x,z)x^\#\times y-z\times\{N(x)y+T(x,y)x^\#\}\qquad\text{(by (17), (19))}\\
&\quad= -T(x,y)\{U_xz+x^\#\times z\}+T(x,z)\{U_xy+x^\#\times y\}\qquad\text{(by (15))}\\
&\quad= -T(x,y)T(x,z)x+T(x,z)T(x,y)x\qquad\text{(by (9))}\\
&\quad= 0.
\end{aligned}$$

Summarizing what these calculations have shown, we have

THEOREM 1 (THE GENERAL CONSTRUCTION). *If the cubic form N, adjoint map $\#$, and basepoint c satisfy*

(i) $x^{\#\#}=N(x)x$,

(ii) $T(x^{\#}, y)=\partial_y N|_x$,

(iii) $c\times y=T(y)c-y$,

(iv) $c^{\#}=c$,

(v) $N(c)=1$

under all scalar extensions then $\mathfrak{J}(N, \#, c)$ is a unital Jordan algebra with U-operator $U_x y=T(x, y)x-x^{\#}\times y$.

We will briefly indicate some of the properties of the algebras $\mathfrak{J}(N, \#, c)$. The first is that every element satisfies the equation

$$x^3-T(x)x^2+S(x)x-N(x)c = 0 \qquad (S(x) = T(x^{\#})) \tag{20}$$

where x^3 is defined to be $U_x x$ and x^2 to be $U_x c$, since

$$\begin{aligned} x^3-T(x)x^2 &= U_x x-T(x)U_x c = \{T(x, x)x-x^{\#}\times x\}-T(x)\{T(x)x-x^{\#}\times c\} \\ &= \{T(x, x)-T(x)T(x)\}x+x^{\#}\times\{T(x)c-x\} \\ &= -T(x\times x)x+x^{\#}\times(x\times c) && \text{(by (16), (8))} \\ &= N(x)c-T(x^{\#})x && \text{(by (10)).} \end{aligned}$$

Also, since $x^2-T(x)x=U_x c-T(x)x=-x^{\#}\times c=x^{\#}-T(x^{\#})c$ we see that

$$x^{\#}= x^2-T(x)x+S(x)c \tag{21}$$

is the usual adjoint.

If we have two such algebras $\mathfrak{J}(N, \#, c)$ and $\mathfrak{J}(\tilde{N}, \tilde{\#}, \tilde{c})$ then any linear map ϕ from $\mathfrak{X}$ to $\tilde{\mathfrak{X}}$ which preserves the norm, adjoint, and basepoint (in the sense that $\tilde{N}(\phi(x))=N(x)$, $\phi(x)^{\tilde{\#}}=\phi(x^{\#})$, $\tilde{c}=\phi(c)$) is necessarily a homomorphism of Jordan algebras, since the algebra structure is built up from these. Indeed, differentiating the relation $N(x)=\tilde{N}(\phi(x))$ by means of the chain rule gives

$$\begin{aligned} T(x, y) &= -\partial_x\, \partial_y \log N|_c = -\partial_x\, \partial_y \log(\tilde{N}\circ\phi)|_c \\ &= -\partial_{\phi(x)}\, \partial_{\phi(y)} \log \tilde{N}|_{\phi(c)} = \tilde{T}(\phi(x), \phi(y)), \end{aligned}$$

so that

$$\begin{aligned} \phi(U_x y) &= \phi(T(x, y)x-x^{\#}\times y) = T(x, y)\phi(x)-\phi(x^{\#})\,\tilde{\times}\,\phi(y) \\ &= \tilde{T}(\phi(x), \phi(y))\phi(x)-\phi(x)^{\tilde{\#}}\,\tilde{\times}\,\phi(y) = \tilde{U}_{\phi(x)}\phi(y) \end{aligned}$$

and $\phi(c)=\tilde{c}$ establish that ϕ is a homomorphism.

Some of the identities relating N, $\#$, and the U-operators are

$$(U_x y)^{\#} = U_{x^{\#}} y^{\#}, \tag{22}$$

$$N(U_x y)U_x y = N(x)^2 N(y)U_x y, \tag{23}$$

$$U_x x^{\#} = N(x)x, \tag{24}$$

$$U_x(x^{\#})^2 = N(x)^2 c. \tag{25}$$

The first is established by

$$\begin{aligned}
&\{T(x, y)x - x^{\#} \times y\}^{\#} \\
&\quad = T(x, y)^2 x^{\#} - T(x, y)x \times (x^{\#} \times y) + (x^{\#} \times y)^{\#} \\
&\quad = T(x, y)^2 x^{\#} - T(x, y)\{N(x)y + T(x, y)x^{\#}\} \\
&\qquad + \{-x^{\#\#} \times y^{\#} + T(x^{\#\#}, y)y + T(y^{\#}, x^{\#})x^{\#}\} \qquad \text{(by (19), (11))} \\
&\quad = -T(x, y)N(x)y - x^{\#\#} \times y^{\#} + N(x)T(x, y)y + T(x^{\#}, y^{\#})x^{\#} \quad \text{(by (4))} \\
&\quad = T(x^{\#}, y^{\#})x^{\#} - x^{\#\#} \times y^{\#} = U_{x^{\#}} y^{\#}
\end{aligned}$$

and from this (23) follows by

$$N(U_x y)U_x y = (U_x y)^{\#\#} = (U_{x^{\#}} y^{\#})^{\#} = U_{x^{\#\#}} y^{\#\#} = N(x)^2 N(y) U_x y.$$

For (24)

$$U_x x^{\#} = T(x, x^{\#})x - x^{\#} \times x^{\#} = 3N(x)x - 2x^{\#\#} = N(x)x$$

by (13) and (4), while for (25)

$$\begin{aligned}
U_x(x^{\#})^2 &= U_x\{T(x^{\#})x^{\#} - x^{\#\#} \times c\} = T(x^{\#})U_x x^{\#} - N(x)U_x(x \times c) \\
&= T(x^{\#})N(x)x - N(x)\{T(x^{\#})x - N(x)c\} \qquad \text{(by (24), 17))} \\
&= N(x)^2 c.
\end{aligned}$$

These identities allow us to establish

THEOREM 2. *An element x in $\mathfrak{J}(N, \#, c)$ is invertible if and only if $N(x)$ is invertible in Φ, in which case $x^{-1} = N(x)^{-1} x^{\#}$. If u is invertible then the isotope $\mathfrak{J}(N, \#, c)^{(u)}$ is just $\mathfrak{J}(N^{(u)}, \#^{(u)}, c^{(u)})$ for $N^{(u)}(x) = N(u)N(x)$, $x^{\#(u)} = N(u)U_u^{-1} x^{\#}$, $c^{(u)} = u^{-1} = N(u)^{-1} u^{\#}$.*

Proof. Recall [5], [4] that x is invertible with inverse y if $U_x y = x$, $U_x y^2 = c$. If x is invertible with inverse y then by (23)

$$c = N(c)c = N(U_x y^2)U_x y^2 = N(x)^2 N(y^2)c,$$

which implies $N(x)$ is invertible in Φ (actually $N(x)^{-1} = N(y)$). Conversely, if $N(x)$ is invertible then (24) and (25) show x is invertible with inverse $y = N(x)^{-1} x^{\#}$.

Assume u is invertible. Introducing the above $N^{(u)}$, $\#^{(u)}$, $c^{(u)}$ we must set

$$\begin{aligned}
T^{(u)}(x, y) &= -\partial_x \partial_y \log N^{(u)}|_{c^{(u)}} = -\partial_x \partial_y \log N|_{c^{(u)}} \\
&= N(c^{(u)})^{-2}\{(\partial_x N|_{c^{(u)}})(\partial_y N|_{c^{(u)}}) - N(c^{(u)})\, \partial_x \partial_y N|_{c^{(u)}}\} \\
&= N(u)^2\{T(c^{(u)\#}, x)T(c^{(u)\#}, y) - N(u)^{-1} T(x \times y, c^{(u)})\} \\
&= T(u, x)T(u, y) - T(x \times y, u^{\#}) = T(U_u x, y)
\end{aligned}$$

since $N(u^{-1})=N(u)^{-1}$ and $(u^{-1})^{\#}=\{N(u)^{-1}u^{\#}\}^{\#}=N(u)^{-2}u^{\#\#}=N(u)^{-1}u$. Checking the axioms of Theorem 1, we see

$$\text{(i)}\quad x^{\#(u)\#(u)} = N(u)U_u^{-1}\{N(u)U_{u^{-1}}x^{\#}\}^{\#} = N(u)U_u^{-1}\{N(u)^2U_{(u^{-1})^{\#}}x^{\#\#}\} \qquad \text{(by (22))}$$
$$= N(u)^3N(u)^{-2}N(x)U_u^{-1}\{U_ux\} = N(u)N(x)x = N^{(u)}(x)x,$$

$$\text{(ii)}\qquad T^{(u)}(x^{\#(u)}, y) = T(U_u\{N(u)U_u^{-1}x^{\#}\}, y) = N(u)T(x^{\#}, y)$$
$$= N(u)\,\partial_yN|_x = \partial_y\{N(u)N\}|_x = \partial_yN^{(u)}|_x,$$

$$\text{(iii)}\quad c^{(u)}\times^{(u)}y = N(u)U_{u^{-1}}\{u^{-1}\times y\} = N(u)\{T(u^{-1\#}, y)u^{-1}-N(u^{-1})y\} \qquad \text{(by (17))}$$
$$= T(u, y)u^{-1}-y = T(U_uu^{-1}, y)u^{-1}-y = T^{(u)}(c^{(u)}, y)c^{(u)}-y,$$

$$\text{(iv)}\qquad c^{(u)\#(u)} = N(u)U_u^{-1}(u^{-1})^{\#} = U_u^{-1}u = u^{-1} = c^{(u)},$$

$$\text{(v)}\qquad N^{(u)}(c^{(u)}) = N(u)N(u^{-1}) = 1.$$

Thus $\mathfrak{J}(N^{(u)}, \#^{(u)}, c^{(u)})$ is a Jordan algebra. It has the same unit element $c^{(u)}=u^{-1}$ as the isotope $\mathfrak{J}(N, \#, c)^{(u)}$ and the same U-operator $U_x^{(u)}=U_xU_u$ since

$$U_x^{(u)}y = T^{(u)}(x, y)x-x^{\#(u)}\times^{(u)}y = T(x, U_uy)x-N(u)U_u^{-1}\{y\times(N(u)U_u^{-1}x^{\#})\}$$
$$= T(x, U_uy)x-U_{N(u)u^{-1}}\{y\times U_u^{-1}x^{\#}\}$$
$$= T(x, U_uy)x-U_{u^{\#}}\{y\times U_u^{-1}x^{\#}\} \qquad \text{(by first part of Theorem)}$$
$$= T(x, U_uy)x-(U_uy)\times U_u(U_u^{-1}x^{\#}) \qquad \text{(by linearized (22))}$$
$$= T(x, U_uy)x-x^{\#}\times U_uy = U_x(U_uy).$$

Since $\mathfrak{J}(N^{(u)}, \#^{(u)}, c^{(u)})$ and $\mathfrak{J}(N, \#, c)^{(u)}$ have the same unit and the same U-operators, they are the same Jordan algebra.

REMARK. It seems to be difficult to decide whether the formulas

$$N(U_xy) = N(x)^2N(y), \qquad N(x^{\#}) = N(x)^2$$

hold in general, although they do if Φ is a field (or more generally if Φ is an integral domain) using (23). The first actually follows from the second, since linearizing the latter gives

$$N(x\times y)+N(x)N(y) = T(x^{\#}, y)T(x, y^{\#})$$

and this can be used to expand $N(U_xy)=N(T(x, y)x-x^{\#}\times y)$.

3. **The Freudenthal construction.** We use the foregoing to construct certain Jordan matrix algebras. Let $\mathfrak{D}$ be a unital alternative ring with *central involution* * (i.e. all *norms* $n(a)=aa^*$ for $a\in\mathfrak{D}$ lie in the center of $\mathfrak{D}$; replacing a by $a+1$ we see all *traces* $t(a)=a+a^*$ also lie in the center). Let Φ be the (commutative associative) subring of *-symmetric elements of the center, and regard $\mathfrak{D}$ as an algebra over Φ. If $\gamma_1, \gamma_2, \gamma_3$ are invertible elements of Φ let $\gamma=\operatorname{diag}\{\gamma_1, \gamma_2, \gamma_3\}$ be the 3×3 matrix with γ_i's down the diagonal. Then $x\to\gamma^{-1}x^{*t}\gamma$ defines an involution on the algebra $\mathfrak{D}_3$ of 3×3 matrices with entries in $\mathfrak{D}$. Let $\mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)$ denote the subspace of

symmetric matrices under this involution whose diagonal entries lie in Φ. Such an element has the form

$$x = \sum_{i=1}^{3} \alpha_i e_i + \sum_{i=1}^{3} a_i[jk] \qquad (\alpha_i \in \Phi,\ a_i \in \mathfrak{D})$$

where (ijk) is a cyclic permutation of (123), $a[ij] = \gamma_j a e_{ij} + \gamma_i a^* e_{ji}$ in terms of the matrix units e_{ij}, and $e_i = e_{ii}$. Following Freudenthal, we apply the construction to

$$\mathfrak{X} = \mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)$$

as a Φ-module, where for x as above and $y = \sum \beta_i e_i + \sum b_i[jk]$ we set

$$\begin{aligned}
N(x) &= \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_2\gamma_3 n(a_1) - \gamma_1\alpha\gamma_3 n(a_2) - \gamma_1\gamma_2\alpha_3 n(a_3) + \gamma_1\gamma_2\gamma_3 t(a_1a_2a_3),\\
T(x, y) &= \sum_i \alpha_i\beta_i + \sum_i \gamma_j\gamma_k t(a_i^*, b_i),\\
x^{\#} &= \sum_i \{\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)\} e_i + \sum_i \{\gamma_i(a_ja_k)^* - \alpha_i a_i\}[jk],\\
c &= e_1 + e_2 + e_3,
\end{aligned}$$

(note $t((a_1a_2)a_3)$ is invariant under cyclic permutations). The $\mathfrak{J}(N, \#, c)$ derived from the above N, $\#$, and c will also be denoted $\mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)$.

Clearly (5), (7), (8) hold: $N(c) = 1$, $c^{\#} = c$,

$$\begin{aligned}
c \times y &= \sum_i (\beta_j + \beta_k) e_i - \sum_i b_i[jk] = \sum_i (\beta_i + \beta_j + \beta_k) e_i - y\\
&= (\beta_1 + \beta_2 + \beta_3) c - y = T(y)c - y.
\end{aligned}$$

For (6),

$$T(x^{\#}, y) = \sum_i \{\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)\}\beta_i + \sum_i \gamma_j\gamma_k t(\gamma_i(a_ja_k) - \alpha_i a_i^*, b_i)$$

while

$$\partial_y N|_x = \sum_i \beta_i\alpha_j\alpha_k - \sum_i \beta_i\gamma_j\gamma_k n(a_i) - \sum_i \alpha_i\gamma_j\gamma_k t(a_i^*, b_i) + \gamma_1\gamma_2\gamma_3 \sum_i t(b_ia_ja_k)$$

since $\partial_b n|_a = t(a^*, b)$. Finally, for (4), let $x^{\#} = \sum \beta_i e_i + \sum b_i[jk]$ and $x^{\#\#} = \sum \delta_i e_i + \sum d_i[jk]$. Then

$$\begin{aligned}
\delta_i &= \beta_j\beta_k - \gamma_j\gamma_k n(b_i)\\
&= \{\alpha_k\alpha_i - \gamma_k\gamma_i n(a_j)\}\{\alpha_i\alpha_j - \gamma_i\gamma_j n(a_k)\} - \gamma_j\gamma_k\{\gamma_i^2 n(a_ja_k) - \alpha_i\gamma_i n(a_i, (a_ja_k)^*) + \alpha_i^2 n(a_i)\}\\
&= \alpha_i\{\alpha_i\alpha_j\alpha_k - \gamma_i\gamma_j\alpha_k n(a_k) - \gamma_i\gamma_k\alpha_j n(a_j) - \gamma_j\gamma_k\alpha_i n(a_i) + \gamma_1\gamma_2\gamma_3 n(a_i, (a_ja_k)^*)\}\\
&\quad + \gamma_i^2\gamma_j\gamma_k n(a_j)n(a_k) - \gamma_j\gamma_k\gamma_i^2 n(a_ja_k)\\
&= N(x)\alpha_i
\end{aligned}$$

since $n(a, b^*) = ab^{**} + b^*a^* = ab + (ab)^* = t(ab)$ and $n(ab) = n(a)n(b)$ because $\mathfrak{D}$ is alternative and * is central. Also

$$\begin{aligned}
d_i &= \gamma_i b_k^* b_j^* - \beta_i b_i \\
&= \gamma_i\{\gamma_k a_i a_j - \alpha_k a_k^*\}\{\gamma_j a_k a_i - \alpha_j a_j^*\} - \{\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i)\}\{\gamma_i a_k^* a_j^* - \alpha_i a_i\} \\
&= \gamma_i\gamma_j\gamma_k(a_i a_j)(a_k a_i) - \gamma_i\gamma_j\alpha_k a_k^*(a_k a_i) - \gamma_i\gamma_k\alpha_j(a_i a_j)a_j^* + \gamma_i\alpha_j\alpha_k a_k^* a_j^* \\
&\quad - \alpha_j\alpha_k\gamma_i a_k^* a_j^* - \gamma_j\gamma_k\alpha_i n(a_i)a_i + \alpha_j\alpha_k\alpha_i a_i + \gamma_i\gamma_j\gamma_k n(a_i)(a_j a_k)^* \\
&= a_i\{\alpha_i\alpha_j\alpha_k - \gamma_i\gamma_j\alpha_k n(a_k) - \gamma_i\gamma_k\alpha_j n(a_j) - \gamma_j\gamma_k\alpha_i n(a_i) \\
&\qquad\qquad + \gamma_i\gamma_j\gamma_k[(a_j a_k)a_i + a_i^*(a_j a_k)^*]\} \\
&= N(x)a_i
\end{aligned}$$

Using the Moufang identity, $a^*(ab) = (ba)a^* = n(a)b$, and $(a_j a_k)a_i + [(a_j a_k)a_i]^* = t((a_j a_k)a_i)$. This establishes $x^{\#\#} = N(x)x$. Hence $\mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)$ satisfies the axioms (4)–(8) (or (i)-(v) in Theorem 1). The same holds for any scalar extension since if $\Omega \supset \Phi$ then $\mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)_\Omega$ is just $\mathfrak{H}(\mathfrak{D}_\Omega)_3, \Omega, \gamma)$ and $\mathfrak{D}_\Omega$ satisfies the same hypotheses as the original $\mathfrak{D}$.

Thus $\mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)$ is a Jordan algebra. It is easily verified that the Jordan structure introduced is just that of a Jordan matrix algebra in the sense of [5, p. 1075], or in the usual sense [1], [4] if Φ is a field of characteristic $\neq 2$. Unfortunately, this method cannot be applied to the more difficult case of Jordan matrix algebras $\mathfrak{H}(\mathfrak{D}_3, \mathfrak{D}_0, \gamma)$ where the norms of elements of $\mathfrak{D}$ just lie in the nucleus, not necessarily the center. What we have established is

THEOREM 3 (FREUDENTHAL CONSTRUCTION). *If $\mathfrak{D}$ is a unital alternative ring with central involution, Φ the symmetric elements of the center, and $\gamma_1, \gamma_2, \gamma_3$ invertible elements of Φ then the space $\mathfrak{H}(\mathfrak{D}_3, \Phi, \gamma)$ of those 3×3 matrices with coefficients in $\mathfrak{D}$ and diagonal coefficients in Φ which are symmetric under the involution $x \to \gamma^{-1}x^{*t}\gamma$ has the structure of a Jordan matrix algebra.*

The most important alternative algebras with central involutions are the composition algebras, of dimension 1, 2, 4, or 8 over their center Φ (a field). The resulting matrix algebras $\mathfrak{H}(\mathfrak{D}_3, \gamma)$ are precisely the reduced central simple Jordan algebras of degree 3 over Φ. In particular, for $\mathfrak{D}$ an 8-dimensional Cayley algebra we get a reduced exceptional central simple Jordan algebra (and all of them arise in this way for suitable choice of $\mathfrak{D}$ and Φ).

THEOREM 4. *If $\mathfrak{D}$ is a composition algebra over a field Φ, $\gamma_1, \gamma_2, \gamma_3$ nonzero elements of Φ, then $\mathfrak{H}(\mathfrak{D}_3, \gamma)$ is a reduced simple Jordan matrix algebra. The trace form $T(x, y)$ is nondegenerate except when $\mathfrak{D} = \Phi$ is a field of characteristic* 2.

The usual proof for matrix algebras (as in [4]) shows $\mathfrak{H}(\mathfrak{D}_3, \gamma)$ is simple since any composition algebra $\mathfrak{D}$ is *-simple. From the formula for $T(x, y)$ we see $x = \sum \alpha_i e_i + \sum a_i[jk]$ is in the radical if and only if each $\alpha_i = 0$ and each a_i is in the radical of the bilinear form $t(a, b)$ on $\mathfrak{D}$. But the trace form derived from $t(a)=a+a^*$ is nondegenerate except when $\mathfrak{D}=\Phi$ has the identity involution $a^*=a$ and has characteristic 2.

REMARK. If Φ has characteristic 2 then any element in $\mathfrak{H}(\Phi_3, \gamma)$ has the form

$$x = \begin{pmatrix} \alpha_1 & \gamma_1 a_3 & \gamma_1 a_2 \\ \gamma_2 a_3 & \alpha_2 & \gamma_2 a_1 \\ \gamma_3 a_2 & \gamma_3 a_1 & \alpha_3 \end{pmatrix} \qquad (\alpha_i, a_i \in \Phi)$$

with norm $N(x)=\alpha_1\alpha_2\alpha_3-\alpha_1\gamma_2\gamma_3a_1^2-\alpha_2\gamma_1\gamma_3a_2^2-\alpha_3\gamma_1\gamma_2a_3^3=\det x$ and trace form $T(x, y)=\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3$. Hence the radical of the trace form is the set of matrices with zero down the diagonal. That the generic norm of a central simple algebra could be degenerate in characteristic 2 makes the usual approach to the structure theory through generic norms less useful. It also shows that associative bilinear forms are not too useful in dealing with the U-operators in general, for we have just seen that the space orthogonal to an ideal need not itself be an ideal (here the ideal is the whole algebra, the orthogonal space is the radical, but the radical of the trace form is not an ideal).

4. **The Springer construction.** Although we cannot construct all algebras of interest by means of nondegenerate trace forms, the assumption of nondegeneracy simplifies the original construction. Throughout we assume Φ is a *field* and $\mathfrak{X}$ a finite-dimensional vector space over Φ.

If N is a cubic form on $\mathfrak{X}$ and $c\in\mathfrak{X}$ a *basepoint* where $N(c)=1$ then we can form the trace form

$$T(x, y) = -\partial_x\, \partial_y \log N|_c = (\partial_x N|_c)(\partial_y N|_c)-\partial_x\, \partial_y N|_c$$

of N at c. We say N is *nondegenerate* at c if its trace form is nondegenerate. For nondegenerate forms we have a unique quadratic mapping $x\to x^\#$ in $\mathfrak{X}$ defined by $T(x^\#, y)=\partial_y N|_x$. We say a nondegenerate cubic form N and basepoint c are *admissible* if the adjoint identity $x^{\#\#}=N(x)x$ holds under all scalar extensions (which will always be the case if the identity holds over Φ and Φ contains more than four elements). Linearizing the adjoint as usual allows us to introduce the U-operator

$$U_x y = T(x, y)x - x^\# \times y.$$

We denote the triple $(\mathfrak{X}, U, c)$ by $\mathfrak{J}(N, c)$.

THEOREM 5 (SPRINGER CONSTRUCTION). *If the cubic form N and basepoint c are admissible then $\mathfrak{J}(N, c)$ is a Jordan algebra (the Jordan algebra of the admissible cubic form N with basepoint c).*

Proof. We will establish the identities (i)–(v) of Theorem 1. The adjoint identity (i) is satisfied by hypothesis, (ii) is satisfied by definition of $x^{\#}$, and (v) is satisfied since c is a basepoint. Euler's formula for homogeneous functions gives

$$T(c, y) = (\partial_c N|_c)(\partial_y N|_c) - \partial_c\, \partial_y N|_c = 3N(c)\, \partial_y N|_c - 2\partial_y N|_c = \partial_y N|_c = T(c^{\#}, y)$$

for all y, so by nondegeneracy of T we have $c^{\#} = c$, thus (iv). Again $T(x \times y, z) = \partial_x\, \partial_y\, \partial_z N|_c$ is symmetric in all variables, so

$$\begin{aligned} T(x, y \times c) &= T(x \times y, c) = \partial_c\, \partial_y\, \partial_x N|_c = \partial_x\, \partial_y N|_c \qquad \text{(by Euler again)} \\ &= (\partial_x N|_c)(\partial_y N|_c) - T(x, y) = T(x, c)T(y, c) - T(x, y) \\ &= T(x, T(y, c)c - y) \end{aligned}$$

for all x gives $y \times c = T(y, c)c - y$ by nondegeneracy, thus (iii). Since N remains admissible under scalar extensions, (i)–(v) remain valid under scalar extensions, and $\mathfrak{J}(N, c)$ is a Jordan algebra by Theorem 1.

Remark. This can also be proved directly, without recourse to Theorem 1, along the lines of [2], [8]. One establishes the identities

$$\begin{aligned} x^{\#\#} &= N(x)x \quad \text{(and its linearizations)}, \\ T(x \times y, z) &= \partial_x\, \partial_y\, \partial_z N|_c \text{ is symmetric}, \\ T(U_x y, z) &= T(y, U_x z), \\ T([U_x y]^{\#}, w) &= T(U_{x^{\#}} y^{\#}, w), \\ T(U_x y \times U_x z, w) &= T(U_{x^{\#}}(y \times z), w), \\ T(y \times c, w) &= T(T(y, c)c - y, w), \\ T(U_{U(x)y} z, w) &= T(U_x U_y U_x z, w), \\ T(U_x\{yxz\}, w) &= T(\{U_x yzx\}, w) \end{aligned}$$

constantly using the second and third to move factors from one side of the bilinear form to the other. By nondegeneracy, the last three show $\mathfrak{J}(N, c)$ is Jordan.

Examples. We have already seen that the Freudenthal construction gives us admissible norm forms (and hence is a special case of the Springer construction) except for $\mathfrak{H}(\Phi_3, \gamma)$ when Φ has characteristic 2. We now give two more trivial examples of admissible forms.

1. Take $\mathfrak{X} = \Phi e_1 \oplus \Phi e_2 \oplus \Phi e_3$, $c = e_1 + e_2 + e_3$, and for $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ set $N(x) = \alpha_1\alpha_2\alpha_3$. Then for $y = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3$ we have $\partial_y N|_x = \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 = \alpha_1\alpha_2\beta_3$, so

$$\begin{aligned} T(x, y) &= \{\partial_x N|_c\}\{\partial_y N|_c\} - \partial_x\, \partial_y N|_c \\ &= \{\alpha_1 + \alpha_2 + \alpha_3\}\{\beta_1 + \beta_2 + \beta_3\} - \{\beta_1(\alpha_2 + \alpha_3) + \beta_2(\alpha_1 + \alpha_3) + \beta_3(\alpha_1 + \alpha_2)\} \\ &= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3. \end{aligned}$$

Clearly this is nondegenerate, and $\partial_y N|_x = T(x^\#, y)$ for

$$x^\# = \alpha_2\alpha_3 e_1 + \alpha_1\alpha_3 e_2 + \alpha_1\alpha_2 e_3.$$

Since

$$\begin{aligned} x^{\#\#} &= (\alpha_1\alpha_3)(\alpha_1\alpha_2)e_1 + (\alpha_2\alpha_3)(\alpha_1\alpha_2)e_2 + (\alpha_2\alpha_3)(\alpha_1\alpha_3)e_3 \\ &= \alpha_1\alpha_2\alpha_3(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = N(x)x \end{aligned}$$

we see N and c are admissible. It is easily seen that the Jordan algebra $\mathfrak{J}(N, c)$ is just the direct sum $\Phi \oplus \Phi \oplus \Phi$ of three copies of the base field.

2. Take $\mathfrak{X} = \Phi e \oplus \mathfrak{X}_0$ and let Q_0 be a nondegenerate quadratic form on $\mathfrak{X}_0$ and $c_0 \in \mathfrak{X}_0$ satisfies $Q_0(c_0) = 1$. Take $c = e + c_0$, and for $x = \alpha e + x_0$ set $N(x) = \alpha Q_0(x_0)$. If $y = \beta e + y_0$ we have $\partial_y N|_x = \beta Q_0(x_0) + \alpha Q_0(x_0, y_0)$ and hence

$$\begin{aligned} T(x, y) &= \{\alpha + Q_0(c_0, x_0)\}\{\beta + Q_0(c_0, y_0)\} - \{\beta Q_0(c_0, x_0) + \alpha Q_0(c_0, y_0) + Q_0(x_0, y_0)\} \\ &= \alpha\beta + Q_0(Q_0(c_0, x_0)c_0 - x_0, y_0) = \alpha\beta + Q_0(x^*, y_0) \end{aligned}$$

is nondegenerate if Q_0 is. Also $\partial_y N|_x = T(x^\#, y)$ for

$$x^\# = Q_0(x_0)e + \alpha x_0^*$$

and

$$\begin{aligned} x^{\#\#} &= Q_0(\alpha x_0^*)e + Q_0(x_0)(\alpha x_0^*)^* = \alpha^2 Q_0(x_0)e + Q_0(x_0)\alpha x_0 \\ &= \alpha Q(x_0)\{\alpha e + x_0\} = N(x)x \end{aligned}$$

and $\mathfrak{J}(N, c)$ is a Jordan algebra. Clearly it is just $\Phi \oplus \mathfrak{J}(Q_0, c_0)$, a copy of the field plus the Jordan algebra of the quadratic form Q_0 with basepoint c_0.

PROPERTIES. From (20) and nondegeneracy we can conclude N is the generic norm (at least if $\dim \mathfrak{J} > 2$) since if $\mathfrak{J}(N, c)$ satisfied a polynomial $x^2 - P(x)x + Q(x)c = 0$ identically then $x^\# = x^2 - T(x)x + S(x)c = F(x)x + G(x)c$ where $F(x) = P(x) - T(x)$ is linear and $G(x) = S(x) - Q(x)$ is quadratic. Then

$$N(x)x = x^{\#\#} = \{F(x)^3 - F(x)G(x)\}x + \{F(x)^2G(x) + F(x)G(x)T(x) + G(x)^2\}c.$$

If $\dim \mathfrak{J} > 1$ we have

$$N(x) = F(x)\{F(x)^2 - G(x)\} \quad \text{and} \quad G(x)\{F(x)^2 + F(x)T(x) + G(x)\} = 0.$$

If $G = 0$ then we have

(i) $N(x) = F(x)^3$,

while if $G \neq 0$ then $F(x)^2 + F(x)T(x) + G(x) = 0$ and

(ii) $N(x) = F(x)\{2F(x)^2 + F(x)T(x)\} = F(x)^2\{2F(x) + T(x)\}$.

Thus we have $N(x) = A(x)^2B(x)$ for A and B linear where in case (i) $A = B = F$ and in case (ii) $A = -F$, $B = 2F + T$. Since $c^\# = c$ we have $F(c) + G(c) = 1$. In case (i) $G(c) = 0$, so $A(c) = B(c) = F(c) = 1$. In case (ii) $F^2 + FT + G = 0$, so $0 = F(c)^2 + 3F(c) + 1 - F(c) = \{F(c) + 1\}^2$ and hence $F(c) = -1$, $A(c) = -F(c) = 1$, $B(c) = 2F(c) + T(c) = 1$. Thus if $\mathfrak{J}$ satisfies $x^2 - P(x)x + Q(x)c = 0$ identically then $N(x) = A(x)^2B(x)$ for linear functions A, B with $A(c)=B(c)=1$, and from this $T(x, y)=B(x)B(y)+2A(x)A(y)$ would be degenerate if $\dim \mathfrak{J}>2$.

If N and c are admissible and $N(u)\neq 0$, then $N^{(u)}=N(u)N$ and $c^{(u)}=u^{-1}$ are admissible. This follows from Theorem 2; note $N^{(u)}$ is nondegenerate since $T^{(u)}(x, y)=T(U_u x, y)$ is nondegenerate.

Another important property is that $\mathfrak{J}(N, c)$ is a nondegenerate Jacobson ring [5, p. 1079]. It is a Jacobson ring, i.e. has the chain condition on idempotents, since it can't have more than three orthogonal idempotents (being of degree three). It is nondegenerate, i.e. has no absolute zero divisors $U_z=0$ except zero, since if $U_z=0$ then $z^3=z^2=0$ and $N(z)=0$ (by (24)), so (20) implies $S(z)z=0$. Thus $S(z)=T(z^\#)=0$, and taking traces of (21) gives $T(z)^2=0$, so $T(z)=0$ also and (21) reduces to $z^\#=0$. Then $U_z x=T(z, x)z-z^\# \times x$ implies $T(z, x)=0$ for all x, and $z=0$ by nondegeneracy.

If we have a pair of admissible forms N, c and $\tilde{N}$, $\tilde{c}$ then any linear *surjection* ϕ from $\mathfrak{X}$ to $\tilde{\mathfrak{X}}$ preserving the norms and units is a homomorphism of Jordan algebras (actually an isomorphism). Indeed, from $\tilde{N}(\phi(x))=N(x)$ and $\phi(c)=\tilde{c}$ we get $\tilde{T}(\phi(x), \phi(y))=T(x, y)$ as in the general construction (so in particular $\phi(x)=0$ implies x is in the radical of T, hence $x=0$ by nondegeneracy, and thus ϕ is necessarily *injective*), from which we get

$$\tilde{T}(\phi(x^\#), \phi(y)) = T(x^\#, y) = \partial_y N|_x = \partial_y(\tilde{N}\circ\phi)|_x = \partial_{\phi(y)}\tilde{N}|_{\phi(x)} = \tilde{T}(\phi(x)^{\tilde{\#}}, \phi(y)).$$

This holds for all y, and since the $\phi(y)$ run through all of $\tilde{\mathfrak{X}}$ by surjectivity we see $\phi(x^\#)=\phi(x)^{\tilde{\#}}$ by nondegeneracy. Thus ϕ preserves norms, adjoints, and units, hence by the general result is an algebra homomorphism (actually an isomorphism, since it is bijective).

5. **The Tits constructions.** Let $\mathfrak{A}$ be a separable associative algebra with 1 of degree 3 over a field Φ (so its generic norm n is of degree 3 and its trace form $t(a, b)=-\partial_a\,\partial_b \log n|_1$ is a nondegenerate symmetric bilinear form; see [7, p. 938] and [4]). We will use the Springer construction to construct a Jordan algebra out of $\mathfrak{A}$.

Let $\mathfrak{X}=\mathfrak{A}_0\oplus\mathfrak{A}_1\oplus\mathfrak{A}_2$ where the $\mathfrak{A}_i$ are copies of $\mathfrak{A}$, and set $c=1\oplus 0\oplus 0=(1, 0, 0)$. Choosing $\mu\neq 0$ from Φ we define, for $x=(a_0, a_1, a_2)$

$$N(x) = n(a_0)+\mu n(a_1)+\mu^{-1}n(a_2)-t(a_0a_1a_2).$$

We denote $\mathfrak{J}(N, c)$ by $\mathfrak{J}(\mathfrak{A}, \mu)$ to reveal the origin of N and c. We will show N and c are admissible.

THEOREM 6 (TITS' FIRST CONSTRUCTION). *If $\mathfrak{A}$ is a separable associative algebra of degree 3 over Φ and μ is a nonzero element of Φ then $\mathfrak{J}(\mathfrak{A}, \mu)$ is a Jordan algebra. $\mathfrak{J}(\mathfrak{A}, \mu)$ is a Jordan division algebra if and only if $\mathfrak{A}$ is an associative division algebra and μ is not the generic norm of an element of $\mathfrak{A}$.*

Proof. We first check N and c are admissible. For $y=(b_0, b_1, b_2)$ we have

$$\partial_y N|_x = t(a_0^\#, b_0)+\mu t(a_1^\#, b_1)+\mu^{-1}t(a_2^\#, b_2)-t(b_0a_1a_2)-t(a_0b_1a_2)-t(a_0a_1b_2)$$

since $\partial_b n|_a = t(a^\#, b)$ where $a^\#$ also denotes the adjoint in $\mathfrak{A}$. Then

$$\begin{aligned} T(x, y) &= \{\partial_x N|_c\}\{\partial_y N|_c\}-\partial_x\,\partial_y N|_c \\ &= \{t(a_0)\}\{t(b_0)\}-t(1\times a_0, b_0)-t(b_1, a_2)-t(a_1, b_2) \\ &= t(a_0, b_0)+t(a_1, b_2)+t(a_2, b_1) \end{aligned}$$

where $a\times b=(a+b)^\#-a^\#-b^\#$, $a\times 1=t(a)1-a$ in $\mathfrak{A}$ as usual. Clearly T is nondegenerate since t is. Also $\partial_y N|_x = T(x^\#, y)$ for

$$x^\# = (a_0^\#-a_1a_2, \mu^{-1}a_2^\#-a_0a_1, \mu a_1^\#-a_2a_0).$$

If we let $x=(a_0, a_1, a_2)$, $x^\#=(b_0, b_1, b_2)$, $x^{\#\#}=(c_0, c_1, c_2)$ then

$$\begin{aligned} c_0 = b_0^\#-b_1b_2 &= \{a_0^\#-a_1a_2\}^\#-\{\mu^{-1}a_2^\#-a_0a_1\}\{\mu a_1^\#-a_2a_0\} \\ &= a_0^{\#\#}-a_0^\#\times a_1a_2+(a_1a_2)^\#-a_2^\#a_1^\#-a_0(a_1a_2)a_0+\mu a_0(a_1a_1^\#)+\mu^{-1}(a_2^\#a_2)a_0 \\ &= \{n(a_0)+\mu n(a_1)+\mu^{-1}n(a_2)\}a_0-\{a_0^\#\times a_1a_2+a_0(a_1a_2)a_0\} \end{aligned}$$

since $a^{\#\#}=n(a)a$, $a^\#a=n(a)1=aa^\#$, $(ab)^\#=b^\#a^\#$ in $\mathfrak{A}$. Now since $\mathfrak{A}$ is of degree 3 it is easily verified that the analogue of (9) holds in $\mathfrak{A}$: $aba=t(a, b)a-a^\#\times b$. Hence the above becomes

$$= \{n(a_0)+\mu n(a_1)+\mu^{-1}n(a_2)-t(a_0, a_1a_2)\}a_0 = N(x)a_0.$$

Similar arguments show $c_i=N(x)a_i$ for $i=1, 2$, so $x^{\#\#}=N(x)x$. This holds for any extension of Φ, so N and c are admissible, and $\mathfrak{J}(\mathfrak{A}, \mu)$ is a Jordan algebra.

Suppose $\mathfrak{J}(\mathfrak{A}, \mu)$ is a Jordan division algebra; by Theorem 2 this is equivalent to $N(x)\neq 0$ for $x\neq 0$. Clearly $n(a)\neq 0$ for $a\neq 0$ or else $N(x)=0$ for $x=(a, 0, 0)\neq 0$. Thus $\mathfrak{A}$ is an associative division algebra. If $\mu=n(a)$ then $x=(-1, 0, a)$ has $N(x)=n(-1)+\mu^{-1}n(a)=0$, so μ can not be a generic norm.

Now suppose $\mathfrak{A}$ is a division algebra and μ is not a generic norm. If $N(x)=0$ for $x\neq 0$ then $x^{\#\#}=N(x)x=0$; either $x^\#=0$ for $x\neq 0$ or else $y=x^\#\neq 0$ has $y^\#=0$. In any case there exists $x=(a_0, a_1, a_2)\neq 0$ with $x^\#=0$. Then $a_0^\#=a_1a_2$, $\mu^{-1}a_2^\#=a_0a_1$, $\mu a_1^\#=a_2a_0$. Since $aa^\#=a^\#a=n(a)1$ in $\mathfrak{A}$, $n(a_0)1=a_0a_1a_2=\mu^{-1}n(a_2)1=a_2a_0a_1=\mu n(a_1)1$. Not all $a_i=0$, so not all $n(a_i)=0$ (since $\mathfrak{A}$ is a division algebra), so none are, and $\mu=n(a_0a_1^{-1})$ is a norm, contrary to hypothesis.

This completes the proof.

For Tit's second construction, let $\mathfrak{A}$ be a separable associative algebra with 1 of degree 3 over a field Ω, * an involution of second kind on $\mathfrak{A}$ with fixed field Φ. Then the space $\mathfrak{A}_0=\mathfrak{H}(\mathfrak{A}, *)$ of symmetric elements contains an element u whose generic norm is of the form $n(u)=\mu\mu^*$ for some nonzero $\mu\in\Omega$. We set $\mathfrak{X}=\mathfrak{A}_0\oplus\mathfrak{A}$ as a vector space over Φ, let $c=(1, 0)$, and for $x=(a_0, a)$ define

$$N(x) = n(a_0)+\mu n(a)+\mu^* n(a^*)-t(a_0aua^*).$$

Note that the values of N actually lie in Φ. Indeed, if $\mathfrak{A}$ is any generically algebraic algebra over Ω and S a semilinear automorphism or antiautomorphism of $\mathfrak{A}$ with associated automorphism σ of Ω then the generic trace and norm satisfy $t(Sx) = t(x)^\sigma$ and $n(Sx) = n(x)^\sigma$. Taking $S = \sigma = *$ in our case we see that $t(a^*) = t(a)^*$ and $n(a^*) = n(a)^*$. Since a_0 and aua^* lie in $\mathfrak{H}(\mathfrak{A}, *)$ we see $n(a_0)$ and $t(a_0, aua^*)$ lie in $\mathfrak{H}(\Omega, *) = \Phi$, as does $\mu n(a) + \mu^* n(a^*)$, so $N(x) \in \Phi$. We denote $\mathfrak{J}(N, c)$ by

$$\mathfrak{J}(\mathfrak{A}, u, \mu, *).$$

THEOREM 7 (TITS' SECOND CONSTRUCTION). *If $\mathfrak{A}$ is a separable associative algebra with involution $*$ of the second kind, and u a symmetric element with norm $n(u) = \mu\mu^*$ for nonzero μ in Ω, then $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ is a Jordan algebra. It is a Jordan division algebra if and only if $\mathfrak{A}$ is an associative division algebra and μ is not the generic norm of an element of $\mathfrak{A}$.*

Proof. The fact that N and c are admissible can be proved as in Theorem 6, using the formulas

$$T(x, y) = t(a_0, b_0) + t(ua^*, b) + t(au, b^*),$$
$$x^\# = (a_0^\# - aua^*, \mu^*(a^*)^\# u^{-1} - a_0 a)$$

for $x = (a_0, a)$ and $y = (b_0, b)$ with a_0, b_0 in $\mathfrak{A}_0$ and a, b in $\mathfrak{A}$. The computations are much the same as in Theorem 6, though at one point we need the relation $\mu\mu^* u^{-1} = n(u)u^{-1} = u^\#$, which follows from our choice of u and μ. We will shortly given an alternate, and more illuminating, proof that $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ is a Jordan algebra.

If $\mathfrak{J}$ is a division algebra then so is $\mathfrak{A}$, since $n(a) = 0$ for $a \neq 0$ would imply $N(x) = 0$ for $x = (0, a) \neq 0$. Also, μ cannot be a generic norm, since $\mu = n(a)$ would imply $x = (a^* u^{-1} a, a^{-1}) \neq 0$ has

$$\begin{aligned} N(x) &= n(a)^* n(u)^{-1} n(a) + \mu n(a)^{-1} + \mu^* n(a)^{*-1} - t(a^* u^{-1} a, a^{-1} u a^{-1*}) \\ &= \mu^*(\mu\mu^*)^{-1}\mu + \mu\mu^{-1} + \mu^*\mu^{*-1} - t(1) = 3 - 3 = 0. \end{aligned}$$

Conversely, if $\mathfrak{A}$ is a division algebra and μ is not a generic norm then $\mathfrak{J}$ is a division algebra, for if $N(x) = 0$ for some $x \neq 0$ then as before $x^\# = 0$ for some $x \neq 0$. For $x = (a_0, a)$ this means $a_0^\# = aua^*$, $\mu^*(a^*)^\# u^{-1} = a_0 a$. Then

$$n(a_0)1 = a_0 a_0^\# = a_0 aua^* = \{\mu^*(a^*)^\# u^{-1}\} ua^* = \mu^*(a^*)^\# a^* = \mu^* n(a^*),$$

or $\mu n(a) = n(a_0)^* = n(a_0)$. If $n(a) = 0$ then $n(a_0) = 0$ too, so $a = a_0 = 0$ since $\mathfrak{A}$ is a division algebra, which contradicts $x \neq 0$. Thus $\mu = n(a_0 a^{-1})$ is a generic norm, which is again a contradiction.

Another way to show that $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ is a Jordan algebra is to show it is imbedded in $\mathfrak{J}(\mathfrak{A}, \mu)$ as the subalgebra of fixed points relative to an automorphism. We claim the involution $*$ on $\mathfrak{A}$ extends to an involution (=automorphism of period 2)

$$x = (a_0, a_1, ua_2) \to x^* = (a_0^*, a_2^*, ua_1^*)$$

on $\mathfrak{J}(\mathfrak{A}, \mu)=\mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$. Clearly this * is a semilinear bijection with associated field automorphism *, and $x^{**}=x$. To see that * preserves the U-structure and unit it suffices to see $N(x^*)=N(x)^*$ and $c^*=c$, since the U-operators were constructed from N and c. But $c=(1, 0, 0)=c^*$ and

$$N(x^*) = n(a_0^*)+\mu n(a_2^*)+\mu^{-1}n(ua_1^*) = n(a_0)^*+\mu^* n(a_1)^*+\mu^{*-1}n(ua_2)^* = N(x)^*.$$

Hence the *-symmetric elements form a Φ-subalgebra $\mathfrak{H}=\mathfrak{H}(\mathfrak{J}(\mathfrak{A}, \mu), *)$ of the Ω-algebra $\mathfrak{J}(\mathfrak{A}, \mu)$. $\mathfrak{H}$ consists of the elements (a_0, a, ua^*) for $a_0 \in \mathfrak{A}_0$, $a \in \mathfrak{A}$. Thus we have a vector-space isomorphism of $\mathfrak{J}(\mathfrak{A}, u, \mu, *)=\mathfrak{A}_0 \oplus \mathfrak{A}$ onto $\mathfrak{H}$ by $(a_0, a) \to (a_0, a, ua^*)$. To see that this is an algebra isomorphism we again need only see that norms and identities are preserved. But $c=(1, 0) \to (1, 0, 0)=c$ and

$$\begin{aligned} N(a_0, a) &= n(a_0)+\mu n(a)+\mu^* n(a^*)-t(a_0, aua^*) \\ &= n(a_0)+\mu n(a)+\mu^{-1}n(ua^*)-t(a_0a(ua^*)) = N(a_0, a, ua^*). \end{aligned}$$

Hence we may identify $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ with the subalgebra of $\mathfrak{J}(\mathfrak{A}, \mu)$ of symmetric elements relative to the involution *.

As a corollary of this, note that $\mathfrak{J}(\mathfrak{A}, u, \mu, *)_\Omega$ is isomorphic to $\mathfrak{J}(\mathfrak{A}, \mu)$ as Ω-algebras. In fact, $\mathfrak{J}(\mathfrak{A}, u, \mu, *)_\Omega \cong \mathfrak{H}_\Omega \cong \Omega\mathfrak{H} \subset \mathfrak{J}(\mathfrak{A}, \mu)$ and $\Omega\mathfrak{H}=\mathfrak{J}(\mathfrak{A}, \mu)$ since if $\alpha \neq \alpha^*$ in Ω then any x can be written

$$x = \{x-\alpha(x-x^*)/(\alpha-\alpha^*)\}+\alpha\{(x-x^*)/(\alpha-\alpha^*)\} \in \mathfrak{H}+\alpha\mathfrak{H} = \Omega\mathfrak{H}.$$

Finally, we remark that if we take $\mathfrak{A}$ to be 9-dimensional central simple, then $\mathfrak{J}(\mathfrak{A}, \mu)$ and $\mathfrak{J}(\mathfrak{A}, u, \mu, *)$ are 27-dimensional exceptional simple Jordan algebras. Tits showed that in characteristic $\neq 2$ one gets all the (finite-dimensional) exceptional simple Jordan algebras by one or the other of his constructions (see [4], [9]). It is not known if this holds for characteristic 2.

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University of Virginia,
Charlottesville, Virginia